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High Order Difference Methods for Heat Equation in Polar Cylindrical Coordinates*

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Fourth-order difference methods for the solution of Poisson equations in cylindrical polar coordinates are proposed. The same technique is then applied to obtain $O(k^2+h^4)$, two level, unconditionally stable ADI methods for the solution of the heat equation in two-dimensional polar coordinates and three-dimensional cylindrical coordinates. Numerical examples given here show that the methods developed here retain their order and accuracy everywhere including the region in the vicinity of the singularity $r=0$.

INTRODUCTION

The numerical solution of the heat equation in polar coordinates is of great importance in problems of heat transfer. In this paper high order difference methods have been proposed to solve these equations. Ciment, Leventhal, and Weinberg [1, 2] have discussed high order operator compact implicit methods for parabolic equations. A symmetrical semi-implicit scheme of $O(k+h^2)$ for the general heat conduction equation was given by Livne and Glasner [3]. Monotone difference schemes for diffusion-convection equations were discussed by Stoyan [4]. Some explicit and implicit schemes for the cylindrical heat equation in one dimension were derived by Mitchell and Pearce [5] and high order difference methods were given by Iyengar and Mittal [6]. An extension of A-Stability to ADI methods was given by Warming and Beam [7]. For problems in polar coordinates, $r=0$, if included in the domain, requires special care. The solutions usually deteriorate in the neighborhood of this singularity and there are two ways in which $r=0$ is dealt with. One approach is to write the differential equation as $r\to 0$ and construct a suitable difference equation valid at $r=0$. The second approach is to avoid $r=0$ by using a mesh starting at, say $r=h/2$ [8].

Recently, Tan [9] used the Chebyshev polynomial expansion and also a

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combination of spectral and finite difference method to find the solution of three dimensional Poisson and Helmholtz equations in cylindrical coordinates. The numerical results in [9] for the spectral finite difference method exhibit second-order accuracy. A second-order method for the solution of the Dirichlet problem of two-dimensional Laplace equation in cylindrical coordinates in an annulus $a \leqslant r \leqslant 1$ by an extrapolated ADI method was given by Evans and Avdelas [10].

For the derivation of the difference schemes of high order we follow the ideas first proposed by Young and Dauwalder [11] for general elliptic equations. A simple procedure to obtain such formulas was recently given by Ananthakrishnaiah, Manohar, and Stephenson [12]. In this paper, we refine this procedure in such a way that the solutions retain the order and accuracy even in the vicinity of the singularity. This idea is used in Section 2 to derive fourth-order difference methods for the solution of three-dimensional Poisson equation in cylindrical coordinates. A separate difference equation of fourth order is derived for $r = 0$ and methods to deal with boundary conditions of mixed type are also discussed. In Section 3, we consider the extension of this procedure to the time-dependent heat equation with variable coefficients. A similar approach was successfully applied by the authors to solve one-dimensional parabolic heat equations [13]. The same approach is then extended to derive two-level difference schemes of $O(k^2 + h^4)$ for the solution of the time-dependent heat equation in two-dimensional cylindrical and spherical coordinates and in three-dimensional cylindrical coordinates. A further refinement allows us to obtain unconditionally stable ADI methods which are of $O(k^2 + h^4)$. It may be mentioned here that no $O(k^2 + h^4)$, two-level ADI methods are known for these problems. It is shown here that for a fixed $\lambda = k/h^2$, these methods are of $O(h^4)$. Also, such ADI methods require the solution of only tridiagonal systems of equations parallel to coordinate axes, at each time step, independent of the order of the method. Thus the amount of computational effort required to solve a problem using the present $O(k^2 + h^4)$ methods is marginally greater than is required by the lower order methods. The additional computation required is in the evaluation of the coefficients at each mesh point. All the computations are performed in double precision on VAX 8600, at the University of Saskatchewan.

2. Difference Schemes for the Poisson Equation

Let us first consider a procedure to construct high order difference schemes for the equation

$$u_{xx} + C(x)\, u_{yy} + D(x)\, u_x + F(x)\, u = G(x, y). \tag{2.1}$$

Consider a uniform mesh $x_i = x_0 + ih$, $y_j = y_0 + jh$, $i = O(1)\, N_1$ and $j = O(1)\, N_2$ and write the Taylor series expansion for the coefficient functions C, D, F, and G about a nodal point (x_i, y_j) which can be taken as the origin in the local coordinates. Let these expansions be

$$C(x)=\sum C_i x^i, \qquad D(x)=\sum D_i x^i, \qquad F(x)=\sum F_i x^i$$

and

$$G(x,y)=\sum p_{i,j} x^i y^j, \qquad u(x,y)=\sum q_{i,j}\, x^i y^j, \qquad i,j=0,1,2,\ldots . \tag{2.2}$$

In order to obtain a difference scheme of order four we assume $q_{i,j}=0$ for $i+j>4$ and similarly for all the other coefficients in (2.2). Substituting (2.2) in (2.1) and comparing the coefficients of $x^i y^j$ for different values of i and j we get

$$\begin{aligned} p_{i,j} = (i+2)(i+1)\, q_{i+2,j} + \sum_{r=0}^{i} [(j+2)(j+1)\, C_r q_{i-r,j+2} \\ + (i+1-r)\, D_r q_{i+1-r,j} + F_r q_{i-r,j}]. \end{aligned} \tag{2.3}$$

A linear combination of equations in (2.3) as in [12] gives

$$\begin{aligned} & h^2[s_0 p_{00} + s_1(p_{10}h + p_{30}h^3) + s_2(p_{01}h + p_{03}h^3) + S_3(p_{20}h^2 + p_{40}h^4) \\ &\quad + s_4(p_{02}h^2 + p_{04}h^4) + s_5(p_{11}h^2 + p_{13}h^4 + p_{31}h^4) \\ &\quad + s_6\, p_{12}h^3 + s_7\, p_{21}h^3 + s_8\, p_{22}h^4] \\ &= [Q_{00}q_{00} + h(Q_{10}q_{10} + Q_{01}q_{01}) + h^2(Q_{20}q_{20} + Q_{11}q_{11} + Q_{02}q_{02}) \\ &\quad + h^3(Q_{30}q_{30} + Q_{21}q_{21} + Q_{12}q_{12} + Q_{03}q_{03}) \\ &\quad + h^4(Q_{40}q_{40} + Q_{31}q_{31} + Q_{22}q_{22} + Q_{13}q_{13} + Q_{04}q_{04})]. \end{aligned} \tag{2.4}$$

Here $s_0, s_1, \ldots, s_8$ are arbitrary parameters which are determined in such a way that Eq. (2.4) results in a difference equation over a 9-point cell for which the difference approximations are well known,

$$\begin{aligned} 2\,(q_{10}h + q_{30}h^3)_{i,j} &= \delta_{2x} u_{i,j} + O(h^5) \\ 2\,(q_{20}h^2 + q_{40}h^4)_{i,j} &= \delta_x^2 u_{i,j} + O(h^6) \\ 4(q_{11}h^2 + q_{13}h^4 + q_{31}h^4)_{i,j} &= \delta_{2x}\delta_{2y} u_{i,j} + O(h^6) \\ 4\,(q_{12}h^3)_{i,j} &= \delta_{2x}\delta_y^2 u_{i,j} + O(h^5) \\ 4\,(q_{22}h^4)_{i,j} &= \delta_x^2\delta_y^2 u_{i,j} + O(h^6), \end{aligned} \tag{2.5}$$

where $\delta_{2x}u_{i,j} = u_{i+1,j} - u_{i-1,j}$ and $\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}$. Similar expressions hold for $(q_{01}h + q_{03}h^3)_{i,j}$, etc. In order to obtain a difference method from (2.4) we select the parameters s_i for $i=1,\ldots,8$ such that the difference approximations (2.5) replace all the Taylor coefficients by differences in (2.4). This requires

$$Q_{10} = Q_{30}, \qquad Q_{01} = Q_{03}, \qquad Q_{20} = Q_{40}, \qquad Q_{02} = Q_{04}, \qquad Q_{11} = Q_{13} = Q_{31}.$$

These equations are sufficient to determine six of the parameters $s_1, ..., s_8$ while the two remaining parameters are arbitrary. The parameter s_0 is chosen to normalize the difference equation. Expressions for $Q_{i,j}$ are given in the Appendix. From the equations $Q_{10} = Q_{30}$ and $Q_{20} = Q_{40}$ we have

$$p_1 s_1 + h p_2 s_3 = h D_0 s_0 \qquad \text{and} \qquad h q_1 s_1 + q_2 s_3 = 2 s_0, \tag{2.6}$$

where $p_1 = 6 + 2D_1 h^2 - (D_3 + F_2) h^4$, $q_1 = 2D_0 - (2D_2 + F_1) h^2$, and $p_2 = 3D_0 + 2D_2 h^2 - (D_4 + F_3) h^4$, $q_2 = 12 + 2D_1 h^2 - (2D_3 + F_2) h^4$. From (2.6), we get

$$s_0 = p_1 q_2 - p_2 q_1 h^2, \qquad s_1 = (D_0 q_2 - 2p_2) h, \qquad s_3 = 2p_1 - D_0 q_1 h^2. \tag{2.7}$$

Either from Eqs. (2.7) or by matching the terms up to $O(h^4)$ in the equations, we get the following approximations to s_0, s_1, and s_3 given by

$$s_0 = 72, \qquad s_1 = 6hD_0, \qquad s_3 = 12. \tag{2.8}$$

These approximations are the same as given in [12], while the values given in (2.7) are the same as those obtained in [13]. Note that it is sufficient to use approximate values of $Q_{i,j}$ when the expressions given in (2.8) are used for the parameters. These approximate values of $Q_{i,j}$ are enclosed in square brackets, while complete expressions for $Q_{i,j}$ as given in the Appendix are to be used when the expressions in (2.7) are used for the parameters.

From the remaining equations, viz. $Q_{01} = Q_{03}$, $Q_{02} = Q_{04}$, and $Q_{11} = Q_{13} = Q_{31}$, we find that $s_2 = s_5 = s_7 = 0$ while s_4 is given by

$$\begin{aligned} &12C_0 s_4 + (12C_1 - F_1 h^2) s_6 h + (12C_2 - F_2 h^2) s_8 h^2 \\ &\quad = [2C_0 s_0 + 2s_1(C_1 + C_3 h^2) h + 2s_3(C_2 + C_4 h^2) h^2]. \end{aligned} \tag{2.9}$$

Here s_6 and s_8 are arbitrary and s_0, s_1, and s_3 are substituted from (2.7) or from (2.8). Without loss of generality these arbitrary parameters can be set equal to zero. Later on, for the time-dependent problems these parameters will be chosen in such a way that ADI methods can be used. Now, a difference method for Eq. (2.1) can be written as

$$L_2 u_{i,j} = h^2 L_1 G_{i,j}, \tag{2.10}$$

where

$$\begin{aligned} 4L_1 &= 4s_0 + 2s_1 \delta_{2x} + 2s_3 \delta_x^2 + 2s_4 \delta_y^2 + s_6 \delta_{2x} \delta_y^2 + s_8 \delta_x^2 \delta_y^2 \\ 4L_2 &= 4Q_{00} + 2Q_{10} \delta_{2x} + 2Q_{20} \delta_x^2 + 2Q_{02} \delta_y^2 + Q_{12} \delta_{2x} \delta_y^2 + Q_{22} \delta_x^2 \delta_y^2. \end{aligned}$$

We shall call the method (2.10) with parameters given by (2.8) along with the approximate values of $Q_{i,j}$ given in square brackets in the Appendix as "Method 2," while method (2.10) with the parameters given by (2.7) and complete expressions for $Q_{i,j}$ as given in the Appendix as "Method 1." The truncation errors of Method 1 and Method 2 are respectively

$$\mathrm{TE}_1 = (3/10)\, h^6(D_x^6 + C_0 D_y^6 + 3D_0 D_x^5)\, u_{i,j} \tag{2.11}$$

$$\mathrm{TE}_2 = \mathrm{TE}_1 + h^6[(1/2)(D_0^2 + 2D_1)\, D_x^4 + 2(D_0 D_1 + 2D_2)\, D_x^3]\, u_{i,j}, \tag{2.12}$$

where $D_x^p = (\partial^p u/\partial x^p)$, etc. Note that the truncation error for Method 1 does not contain lower order derivatives of u while the truncation error of Method 2 does. The presence of the lower order derivatives affect the results in the vicinity of the singularity $r = 0$ particularly for the time-dependent problems when both h and k approach zero. Method 2 is a particular case of the methods proposed in [12], while Method 1 is a generalization of the one-dimensional case considered in [13]. For the mesh points away from the singularity both methods produce results which are comparable.

Following the same procedure, we can obtain Method 1 for the three-dimensional Poisson equation

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_z = f(r, \theta, z). \tag{2.13}$$

For the discretization of (2.13) we consider a uniform mesh $r_i = r_0 + ih$, $\theta_j = \theta_0 + jh$, $z_m = z_0 + mh$, $i = O(1)\, N_1$, $j = O(1)\, N_2$, $m = O(1)\, N_3$. The difference equation for (2.13) is

$$L_2 u_{i,j,m} = h^2 L_1 f_{i,j,m}, \tag{2.14}$$

where $u_{i,j,m} = u(r_i, \theta_j, z_m)$, etc., and

$$4L_1 = 4s_0 + 2s_1\delta_{2r} + 2s_3\delta_r^2 + 2s_5\delta_\theta^2 + (s_0/3)\,\delta_z^2 + s_{13}\delta_{2r}\delta_\theta^2 + s_{14}\delta_{2r}\delta_z^2$$
$$+ s_{17}\delta_r^2\delta_z^2 + s_{19}\delta_r^2\delta_\theta^2 + s_{22}\delta_\theta^2\delta_z^2$$

$$4L_2 = 2Q_{100}\delta_{2r} + 2Q_{200}\delta_r^2 + 2Q_{020}\delta_\theta^2 + 2Q_{002}\delta_z^2 + Q_{120}\delta_{2r}\delta_\theta^2 + Q_{102}\delta_{2r}\delta_z^2$$
$$+ Q_{202}\delta_r^2\delta_z^2 + Q_{220}\delta_r^2\delta_\theta^2 + Q_{022}\delta_\theta^2\delta_z^2$$

$$s_0 = 6(12 - 7p^2 + 5p^4), \quad s_1 = 2p(3 - 3p^2 + 2p^4), \quad s_3 = 2\,(6 - 3p^2 + 2p^4), \quad p = h/r_i$$

$$s_5 = r_2 + 2ps_{13} - 3p^2 s_{19}, \quad r_2 = [s_0 - 2p(1 + 2p^2)\, s_1 + p^2(3 + 5p^2)\, s_3]/6,$$

$$Q_{100} = ps_0 - p^2(1 + p^2)(s_1 - ps_3), \quad Q_{002} = 2s_0, \quad Q_{200} = 2s_0 + 2p(1 + p^2)(s_1 - ps_3),$$

$$Q_{020} = 12r_2/r_i^2, \quad Q_{120} = (2/r_i^2)[(1 + 3p^2)\, s_1 - 2p(1 + 2p^2)\, s_3] + pr_2 + p^2 s_{13} - 2p^3 s_{19},$$

$$Q_{102} = 2s_1 + ps_6 - p^2 s_{14} + p^3 s_{17}, \quad Q_{202} = 2s_3 + (s_0/3) + 2ps_{14} - 2p^2 s_{17},$$

$$Q_{022} = 2r_2 + 4ps_{13} - 6p^2 s_{19} + (2/r_i^2)[(s_0/6) - 2ps_{14} + 3p^2 s_{17}],$$

$$Q_{220} = 2(r_2 + 3ps_{13} - 4p^2 s_{19}) + (2/r_i^2)(s_3 - 2ps_1 + 3p^2 s_3).$$

The method (2.14) is a 19-point formula. The parameters s_{13}, s_{14}, s_{17}, s_{19}, and s_{22} are arbitrary and may be set equal to zero.

If $r=0$ is an interior point or a boundary point such that the solution is to be obtained at this point then we need a difference equation which is valid at $r=0$. As $r \to 0$, we get from (2.13),

$$2u_{rr} + (1/2)\, u_{rr\theta\theta} + u_{zz} = f(0, \theta, z). \tag{2.15}$$

The corresponding difference equation at $r=0$ is

$$(h^2 L_3 + L_4)\, u_{0,j,m} = h^4 L_5 f_{0,j,m}, \tag{2.16}$$

where

$$L_3 = 24\delta_r^2 + 12\delta_z^2 + 3\delta_r^2\delta_z^2 + 2\delta_r^2\delta_\theta^2 + \delta_\theta^2\delta_z^2$$
$$L_4 = 6\delta_r^2\delta_\theta^2 + (1/2)\,\delta_r^2\delta_\theta^2\delta_z^2 \qquad \text{and} \qquad L_5 = 12 + \delta_r^2 + \delta_\theta^2 + \delta_z^2.$$

For $r=0$, $u_r = u_{\theta\theta} = u_{r\theta\theta} = 0$, and the difference equation (2.16) reduces to

$$\begin{aligned} h^2(-48 + 6\delta_z^2)\, u_{0,j,m} &+ [h^2(48 + 6\delta_z^2) + 4(h^2+3)\,\delta_\theta^2 + \delta_\theta^2\delta_z^2]\, u_{1,j,m} \\ &= h^4[12 + \delta_r^2 + \delta_\theta^2 + \delta_z^2] f_{0,j,m}. \end{aligned} \tag{2.17}$$

Mixed Boundary Condition

Let a typical boundary condition be

$$\alpha(\theta, z)\, u + \beta(\theta, z)\, u_r = \gamma(\theta, z) \qquad \text{at} \quad r = r_1. \tag{2.18}$$

Using the fourth-order approximation $u_r \approx (\delta_{2r}/2h(1 + \delta_r^2/6))\, u$ in (2.18), we get

$$(h\alpha_{j,m} - 3\beta_{j,m})\, u_{i-1,j,m} + 4\alpha_{j,m} h u_{i,j,m} + (h\alpha_{j,m} + 3\beta_{j,m})\, u_{i+1,j,m} = 6h\gamma_{j,m}. \tag{2.19}$$

We use both the differential equation and the boundary condition (2.18) to get a difference equation valid at a boundary point. The external points that are introduced are eliminated by combining the difference equations (2.10) and (2.19). We now apply the above methods to the following two problems.

EXAMPLE 1. The problem [9] is to solve the Poisson's equation (2.13) with the exact solution

$$u(r, \theta, z) = (\cos r^* + \sin r^*)(\cos \theta_1 + \sin \theta_1)(\cos z_1 + \sin z_1),$$

where $2 \leqslant r \leqslant 4$, $0 \leqslant \theta \leqslant 0.5$, $-1 \leqslant z \leqslant 1$, and $r^* = \pi(r-4)/2$, $\theta_1 = \pi(2\theta - 1)$, $z_1 = \pi(z-1)/2$. The right-hand side $f(r, \theta, z)$ is obtained by substituting u in (2.13). Dirichlet conditions are used on the boundary.

TABLE I

Maximum Absolute Errors in Solution

h	Problem 1	Problem 2
1/8	1.051(−3)	2.228(−4)
1/16	4.241(−5)	1.371(−5)

EXAMPLE 2. The problem is to solve the Poisson's equation (2.13) with the exact solution

$$u(r, \theta, z) = r^5 (\cos \theta_1 + \sin \theta_1)(\cos z_1 + \sin z_1),$$

where $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 0.5$, $-0.5 \leqslant z \leqslant 0.5$, $\theta_1 = \pi(2\theta - 1)$, and $z_1 = \pi(2z - 1)/2$. The right-hand side $f(r, \theta, z)$ is obtained from the differential equation. Dirichlet conditions are used on the boundary.

Both the problems are solved with $h = \frac{1}{8}$ and $\frac{1}{16}$ giving system of equations of orders $15 \times 3 \times 15$, $31 \times 7 \times 31$; $7 \times 3 \times 7$, $15 \times 7 \times 15$ in Problems 1 and 2, respectively. These systems are solved by an SOR procedure. The maximum absolute errors are given in Table I. Our results display fourth-order accuracy everywhere including the region near $r = 0$. For example, in Problem 2, on $r = \frac{1}{8}$ the maximum absolute errors obtained with $h = \frac{1}{8}$ and $\frac{1}{16}$ are 2.23(−5) and 1.49(−7), respectively. Computational results of Tan [9, Table 3] show that the spectral finite difference method (using second-order differencing in the θ direction) produces results of second order.

3. TIME-DEPENDENT HEAT EQUATION

Consider now the two-dimensional heat equation

$$u_t = u_{xx} + D(x)\, u_x + C(x)\, u_{yy} + F(x)\, u. \tag{3.1}$$

Substituting $G_{i,j} = (u_t)_{i,j}$ in (2.10) and using the approximation $\partial/\partial t \approx [\nabla_t / k(1 - 0.5\nabla_t)]$, where k is the mesh length in t and ∇_t is the backward difference operator in t, we obtain the following difference equation for (3.1)

$$(L_1 - 0.5\lambda L_2)\, u_{i,j}^{n+1} = (L_1 + 0.5\lambda L_2)\, u_{i,j}^{n}, \tag{3.2}$$

where $\lambda = k/h^2$ is the mesh ratio parameter and L_1 and L_2 are defined earlier. The truncation error of the method is

$$\mathrm{TE} = [-12k^3 u_{ttt} + (3/10)\, kh^4 (D_x^6 + C_0 D_y^6 + 3D_0 D_x^5)\, u]_{i,j} + \cdots . \tag{3.3}$$

Note that (3.2) requires solution of a system of equations with a large band width at each time level. It is also difficult to study the stability of (3.2). We therefore modify the difference equation (3.2) in such a way that the order of the truncation error is unchanged while the resulting difference equations require the solution of only tridiagonal matrices at each time level. This difference scheme is given by

$$[a_1 + a_2\delta_{2x} + a_3\delta_x^2][1 + c_1\delta_y^2]\, u_{i,j}^{n+1} = [b_1 + b_2\delta_{2x} + b_3\delta_x^2][1 + c_2\delta_y^2]\, u_{i,j}^n, \tag{3.4}$$

where $a_1 = s_0 - 0.5\lambda Q_{00}$, $2a_2 = s_1 - 0.5\lambda Q_{10}$, $2a_3 = s_3 - 0.5\lambda Q_{20}$, $12c_1 = 1 - 6\lambda C(x)$, $b_1 = s_0 + 0.5\lambda Q_{00}$, $2b_2 = s_1 + 0.5\lambda Q_{10}$, $2b_3 = s_3 + 0.5\lambda Q_{20}$, $12c_2 = 1 + 6\lambda C(x)$. This factorization of the difference operator changes only the coefficient of k^3 in the TE (3.3).. Using the von Neumann method, the amplification factor ξ of (3.4) can be written as $\xi = \xi_1\xi_2$, where

$$\xi_1 = \{[b_1 - 2b_3(1 - \cos\beta h)] + 2b_2 i \sin\beta h\}/\{[a_1 - 2a_3(1 - \cos\beta h)] + 2a_2 i \sin\beta h\} \tag{3.5}$$

$$\xi_2 = [1 - 2c_2(1 - \cos\gamma h)]/[1 - 2c_1(1 - \cos\gamma h)]. \tag{3.6}$$

It is seen that $|\xi_2| \leqslant 1$ for $C(x) \geqslant 0$ and hence for stability we require $|\xi_1| \leqslant 1$. A split form of (3.4) is

$$[a_1 + a_2\delta_{2x} + a_3\delta_x^2]\, v_{i,j} = [b_1 + b_2\delta_{2x} + b_3\delta_x^2][1 + c_2\delta_y^2]\, u_{i,j}^n \tag{3.7}$$

$$[1 + c_1\delta_y^2]\, u_{i,j}^{n+1} = v_{i,j}. \tag{3.8}$$

The intermediate boundary conditions required for the solution of $v_{i,j}$ are obtained from (3.8).

Consider now, the two-dimensional problem

$$u_t = u_{rr} + \alpha r^{-1} u_r + r^{-2} u_{\theta\theta} + eu, \qquad e \text{ constant}, \tag{3.9}$$

where $0 < \alpha < 1$ or $\alpha = 1, 2$. When $\alpha = 1, 2$ it is a two-dimensional equation in cylindrical and spherical coordinates, respectively. Replacing the variables x, y by r, θ, respectively, and setting $D = \alpha/r$, $C = 1/r^2$, $F = e$ in (3.4), we get the ADI method in split form as (3.7) and (3.8), where

$$\begin{aligned} &s_0 = 6[12 - \alpha(6+\alpha)p^2 + \alpha(4+\alpha)p^4], \qquad s_1 = 2\alpha p[3 - (2+\alpha)p^2 + (1+\alpha)p^4] \\ &s_3 = 2[6 - \alpha(2+\alpha)p^2 + \alpha(1+\alpha)p^4], \qquad p = h/r_i, \qquad C(r) = r_i^{-2}, \\ &Q_{00} = eh^2 s_0, \qquad Q_{10} = Q_{10}^* + eh^2 s_1, \qquad Q_{20} = Q_{20}^* + eh^2 s_3, \\ &Q_{10}^* = \alpha p[s_0 - p(1+p^2)(s_1 - ps_3)], \qquad Q_{20}^* = 2[s_0 + \alpha p(1+p^2)(s_1 - ps_3)]. \end{aligned} \tag{3.10}$$

Note that the values of s_0, s_1, s_3, Q_{00}, Q_{10}, and Q_{20} are the same as were obtained in the one-dimensional case [13], where it was shown that $|\xi_1| \leqslant 1$ for all $e \leqslant 0$.

Hence the ADI method (3.7), (3.8) with the coefficients given in (3.10) for Eq. (3.9) is unconditionally stable for all $e \leqslant 0$. The TE now becomes

$$\mathrm{TE} = k^3 T_1 + kh^4 T_2 + \cdots, \tag{3.11}$$

where $T_1 = 12\,[-u_{ttt} + 3r^{-2}\{D_r^2 - (4-\alpha)\ r^{-1}D_r + 2(3-\alpha)\,r^{-2} + e\}\ D_\theta^2 u_t]_{i,j}$ and $T_2 = (\frac{3}{10})[D_r^6 + 3\alpha r_i^{-1} D_r^5 + r_i^{-2} D_\theta^6]\ u_{i,j}$.
For the two-dimensional problem

$$u_t = u_{rr} + \alpha r^{-1} u_r + u_{zz} + eu, \qquad e \text{ constant}, \tag{3.12}$$

we have the ADI method (3.7), (3.8) with $D = \alpha/r$, $C = 1$, $F = e$, and the variables x, y are replaced by r, z, respectively. The other coefficients are given by (3.10). The method is again unconditionally stable. The TE now has the form

$$\mathrm{TE} = k^3 T_3 + kh^4 T_4 + \cdots, \tag{3.13}$$

where $T_3 = 12\,[-u_{ttt} + 3(D_t - D_z^2)\ D_z^2 u_t]_{i,j}$ and $T_4 = (\frac{3}{10})[D_r^6 + 3\alpha r_i^{-1} D_r^5 + D_z^6]\ u_{i,j}$.
Method (3.4) can be extended to the three-dimensional case,

$$u_t = u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta} + u_{zz} + eu, \qquad e \text{ constant}. \tag{3.14}$$

This extension is equivalent to the ADI formulation of (2.10), which is given by

$$\begin{aligned}[][a_1 + a_2\delta_{2r} + a_3\delta_r^2][1 + c_3\delta_\theta^2][1 + d_1\delta_z^2]\ u_{i,j,m}^{n+1}& \\ = [b_1 + b_2\delta_{2r} + b_3\delta_r^2][1 + c_4\delta_\theta^2][1 + d_2\delta_z^2]\ u_{i,j,m}^n&, \end{aligned}\tag{3.15}$$

where $12c_3 = 1 - (6\lambda/r_i^2)$, $12d_1 = 1 - 6\lambda$, $12c_4 = 1 + (6\lambda/r_i^2)$, $12d_2 = 1 + 6\lambda$, and a_1, a_2, etc., are defined earlier. The split form is

$$[a_1 + a_2\delta_{2r} + a_3\delta_r^2]\ w_{i,j,m} = [b_1 + b_2\delta_{2r} + b_3\delta_r^2][1 + c_4\delta_\theta^2][1 + d_2\delta_z^2]\ u_{i,j,m}^n \tag{3.16}$$

$$[1 + c_3\delta_\theta^2]\ v_{i,j,m} = w_{i,j,m} \tag{3.17}$$

$$[1 + d_1\delta_z^2]\ u_{i,j,m}^{n+1} = v_{i,j,m}. \tag{3.18}$$

The intermediate boundary conditions for the solution of $w_{i,j,m}$ and $v_{i,j,m}$ are obtained from Eqs. (3.17) and (3.18). The TE of (3.15) is

$$\mathrm{TE} = k^3 T_5 + kh^4 T_6 + \cdots, \tag{3.19}$$

where $T_5 = T_3 + 36[r^{-2}(D_r^2 - 3r^{-1}D_r + 4r^{-2} + e)\ D_\theta^2(u_t - D_z^2 u)]_{i,j,m}$ and $T_6 = (\frac{3}{10})(D_r^6 + 3r_i^{-1} D_r^5 + r_i^{-2} D_\theta^6 + D_z^6)\ u_{i,j,m}$. Method (3.15) is also unconditionally stable.

In (3.9), (3.12), and (3.14), we have assumed $e = $ constant, which is often the case. However, it is possible to extend the method to include the cases $e = e(r)$, $e = e(r, z)$, and $e = e(r, \theta)$. These generalizations are now discussed. Suitable changes in the independent variables are assumed.

(a) $e = e(r)$. If $e = e(r)$ in (3.9) or (3.12), then the method (3.7) and (3.8) retains its order, $O(k^2 + h^4)$, with the new definitions

$$\begin{aligned} Q_{00} &= h^2[e_0 s_0 + (e_1 h + e_3 h^3)\, s_1 + (e_2 h^2 + e_4 h^4)\, s_3] \\ Q_{10} &= Q_{10}^* + h^2[(e_0 + e_2 h^2)\, s_1 + (e_1 h + e_3 h^3)\, s_3] \\ Q_{20} &= Q_{20}^* + h^2[e_1 h s_1 + (e_0 + e_2 h^2)\, s_3]. \end{aligned} \tag{3.20}$$

When $e = e(r)$ in (3.14), method (3.15) retains its order and the TE remains the same as given in (3.19).

(b) $e = e(r, z)$. If $e = e(r, z)$, then the method (3.4) is modified so that it retains its order by replacing the second brackets on the left- and right-hand sides by

$$[1 - \lambda^* h^3 e_{01} \delta_{2z} + d_1 \delta_z^2], \qquad [1 + \lambda^* h^3 e_{01} \delta_{2z} + d_2 \delta_z^2], \tag{3.21a}$$

where $\lambda^* = \lambda/24$, $12 d_1 = 1 - 6\lambda$, and $12 d_2 = 1 + 6\lambda$, respectively, and we define

$$\begin{aligned} Q_{00} &= h^2[\{e_{00} + (e_{02} h^2 + e_{04} h^4)/6\}\, s_0 + \{e_{10} h + e_{30} h^3 + (h^3/6)\, e_{12}\}\, s_1 \\ &\quad + \{e_{20} h^2 + e_{40} h^4 + (h^4/6)\, e_{22}\}\, s_3] \\ Q_{10} &= Q_{10}^* + h^2[\{e_{00} + e_{20} h^2 + (h^2/6)\, e_{02}\}\, s_1 + \{e_{10} h + e_{30} h^3 + (h^3/6)\, e_{12}\}\, s_3] \\ Q_{20} &= Q_{20}^* + h^2[e_{10} h s_1 + \{e_{00} + e_{20} h^2 + (h^2/6)\, e_{02}\}\, s_3]. \end{aligned} \tag{3.21b}$$

Note that in the above $e_{10} = \partial e/\partial r$, $e_{20} = (\partial^2 e/\partial r^2)/2$, $e_{12} = (\partial^3 e/\partial r\, \partial z^2)/2$, etc. If $e = e(r, z)$ in (3.14), then the third brackets on the left- and right-hand sides of (3.15) are replaced by (3.21a) with Q_{00}, etc., being defined by (3.21b).

(c) $e = e(r, \theta)$. If $e = e(r, \theta)$ in (3.9) or (3.14), then the method (3.4) or (3.15) is modified so that it retains its order by replacing the second brackets on the left- and right-hand sides by

$$[1 - \lambda^* h^3 e_{01} \delta_{2\theta} + c_3 \delta_\theta^2], \qquad [1 + \lambda^* h^3 e_{01} \delta_{2\theta} + c_4 \delta_\theta^2], \tag{3.22}$$

where c_3 and c_4 are defined in (3.15), respectively. Q_{00}, etc., are again defined by (3.21b).

If $r = 0$ is a part of the boundary and the solution at $r = 0$ is also to be determined then we need a difference equation valid at $r = 0$. We illustrate the procedure for one of the equations, say (3.12) for $\alpha = 1$. As in Section 2, we have, as $r \to 0$,

$$u_t = 2u_{rr} + u_{zz}. \tag{3.23}$$

A suitable $O(k^2 + h^4)$ approximation is

$$(1 + \lambda_1 \delta_r^2)(1 + \lambda_2 \delta_z^2)\, u_{0,j}^{n+1} = (1 + \lambda_3 \delta_r^2)(1 + \lambda_4 \delta_z^2)\, u_{0,j}^n, \tag{3.24}$$

TABLE II

Maximum Absolute Errors. Example 3

		Method 1		Method 2	
λ	$t\backslash h$	0.1	0.05	0.1	0.05
1	0.8	0.244(−5)	0.157(−6)	0.234(−5)	0.148(−6)
	1.6	0.155(−6)	0.997(−8)	0.149(−6)	0.943(−8)
2	0.8	0.103(−4)	0.648(−6)	0.102(−4)	0.638(−6)
	1.6	0.654(−6)	0.412(−7)	0.648(−6)	0.406(−7)
4	0.8	0.418(−4)	0.261(−5)	0.417(−4)	0.260(−5)
	1.6	0.266(−5)	0.166(−6)	0.265(−5)	0.166(−6)

where $12\lambda_1 = 1 - 12\lambda$, $12\lambda_2 = 1 - 6\lambda$, $12\lambda_3 = 1 + 12\lambda$, $12\lambda_4 = 1 + 6\lambda$. A split form is

$$(1 + \lambda_1 \delta_r^2)\, v_{0,j} = (1 + \lambda_3 \delta_r^2)(1 + \lambda_4 \delta_z^2)\, u_{0,j}^n \tag{3.25}$$

$$(1 + \lambda_2 \delta_z^2)\, u_{0,j}^{n+1} = v_{0,j}, \tag{3.26}$$

where, in (3.24) we are to use the condition $u_r = 0$ at $r = 0$. This implies that $v_{-1,j} = v_{1,j}$. Note that Eq. (3.26) is same as (3.8) valid along lines parallel to z-axis. The intermediate boundary conditions are obtained from (3.26).

In all the above methods, the integration is first carried along lines parallel to the r-axis, then along the θ-axis, and then along the z-axis. These methods produce tridiagonal systems for solution along lines parallel to the axes. They are all two-level formulas so that no extra starting values are required and all these methods are unconditionally stable so that large step lengths along the t-direction may be used. All the above methods shall be referred to as "Method 1." As in Section 2, we may define a "Method 2" by choosing s_0, s_1, s_3 as in (2.8) and Q_{00}, Q_{10}, Q_{20} being the quantities inside the square brackets in the Appendix. These methods are also of $O(k^2 + h^4)$. For a fixed λ, all the above methods behave like fourth-order methods.

TABLE III

Maximum Absolute Errors on Initial Time Levels. Example 3

		Method 1			Method 2		
λ	$t\backslash h$	0.1	0.05	ER	0.1	0.05	ER
0.1	0.001	0.285(−6)	0.279(−7)	10.2	0.287(−6)	0.373(−7)	7.7
	0.002	0.454(−6)	0.321(−7)	14.1	0.474(−6)	0.480(−7)	9.9
0.2	0.002	0.464(−6)	0.330(−7)	14.1	0.480(−6)	0.474(−7)	10.1
	0.004	0.614(−6)	0.340(−7)	18.1	0.687(−6)	0.585(−7)	11.7

Note. ER is error ratio $E(0.1)/E(0.05)$.

TABLE IV

Exact Solution (Ex) and absolute error in the numerical solution on $r=0$, Example 3 (Method 1): ER_1 = absolute error with $h=0.1$; ER_2 = absolute error with $h=0.05$

	λ	1.0		4.0	
z	t	0.8	1.6	0.8	1.6
−0.8	Ex	−0.786500(−1)	−0.490907(−2)	−0.786500(−1)	−0.490907(−2)
	ER_1	0.1138(−5)	0.7204(−7)	0.1993(−4)	0.1260(−5)
	ER_2	0.7399(−7)	0.4682(−8)	0.1246(−5)	0.7885(−7)
−0.4	Ex	−0.871839(−1)	−0.544174(−2)	−0.871839(−1)	−0.544174(−2)
	ER_1	0.2356(−5)	0.1497(−6)	0.4060(−4)	0.2578(−5)
	ER_2	0.1514(−6)	0.9620(−8)	0.2538(−5)	0.1612(−6)
0.0	Ex	−0.624168(−1)	−0.389586(−2)	−0.624168(−1)	−0.389586(−2)
	ER_1	0.2176(−5)	0.1391(−6)	0.3732(−4)	0.2383(−5)
	ER_2	0.1396(−6)	0.8920(−8)	0.2333(−5)	0.1491(−6)

EXAMPLE 3. The problem is to solve (3.12) with $\alpha=1$, $e=0$ in the region $0 \leqslant r \leqslant 1$, $-1 \leqslant z \leqslant 1$. The exact solution is $u(r, z, t) = J_0(r)$ $(\cos z_1 + \sin z_1)$ $\exp(-\beta t)$, where $z_1 = \pi\,(z-1)/2$ and $\beta = 1 + \pi^2/4$. At $r=0$, we have $u_r = 0$ and all other conditions are provided from the exact solution.

At $r=0$, we use (3.25) and (3.26). At all other mesh points we use (3.7), (3.8) with (3.10) ($\alpha=1$, $C(r)=1$). The problem is also solved using Method 2. Maximum absolute errors are given in Table II for $\lambda = 1, 2, 4$ with $h=0.1$, 0.05. Table III gives the maximum absolute errors on the first two time levels for $\lambda=0.1$, 0.2. Table IV gives the exact solution and absolute errors on $r=0$ for $\lambda=1, 4$ and at some selected points of $z=0, -0.4, -0.8$. These values of z are chosen, because the maximum error on $r=0$ occurs in the vicinity of $z=-0.4$.

For large t, it is seen that both the Methods 1 and 2 give almost the same results. At the initial time levels for large λ, the maximum absolute errors are almost same, the Method 1 being marginally better. However, at the initial time levels for small λ, Method 2 has larger errors and does not show the required error ratio (ER) as given in Table III. The numerical solutions remain accurate even on $r=0$. It is found that the maximum error generally occurs on $r=h$, except for the first few time levels where it occurs on $r=0$. However, the difference between the errors on $r=0$ and $r=h$ is marginal.

CONCLUSIONS

In this paper we have derived fourth-order difference methods which are applicable to the Poisson equation in cylindrical polar coordinates. These ideas are extended to derive $O(k^2+h^4)$, two-level, unconditionally stable ADI methods for the solution of the multidimensional heat equation in polar coordinates. These methods give accurate results everywhere including the region in the vicinity of $r=0$. A reason for this is that the truncation errors do not contain lower order derivatives. A special treatment is required if $r=0$ is a part of the boundary and the solution is to be determined at $r=0$. This is illustrated for the Eq. (3.12) in the r-z plane. A similar procedure can be invoked for the problems in the r-θ plane. The solutions remain accurate at all points on $r=0$ and the order is preserved. The computational effort required for the use of these higher order ADI methods is about the same as required for the lower order methods. The additional computational effort required is in the evaluation of the coefficients at each mesh point.

Appendix

$$Q_{00}=[h^2s_0F_0+h^3s_1F_1+h^4s_3F_2]+h^5s_1F_3+h^6s_3F_4$$

$$Q_{10}=[hs_0D_0+h^2s_1(D_1+F_0)+h^3s_3(D_2+F_1)] + h^4s_1(D_3+F_2)+h^5s_3(D_4+F_3)$$

$$Q_{01}=[h^2s_2F_0+h^3s_5F_1]+h^4s_7F_2+h^5s_5F_3$$

$$Q_{20}=[2s_0+2hs_1D_0+h^2s_3(2D_1+F_0)]+h^3s_1(2D_2+F_1)+h^4s_3(2D_3+F_2)$$

$$Q_{11}=[hs_2D_0+h^2s_5(D_1+F_0)]+h^3s_7(D_2+F_1)+h^4s_5(D_3+F_2)$$

$$Q_{02}=[2s_0C_0+2hs_1C_1+2h^2s_3C_2+h^2s_4F_0] + 2h^3s_0C_3+h^3s_6F_1+2h^4s_3C_4+h^4s_8F_2$$

$$Q_{30}=[6s_1+3hs_3D_0]+h^2s_1(3D_1+F_0)+h^3s_3(3D_2+F_1)$$

$$Q_{21}=[2s_2+2hs_5D_0]+h^2s_7(2D_1+F_0)+h^3s_5(2D_2+F_1)$$

$$Q_{12}=[2C_0s_1+2hs_3C_1]+hs_4D_0+2h^2s_1C_2 + h^2s_6(D_1+F_0)+2h^3s_3C_3+h^3s_8(D_2+F_1)$$

$$Q_{03}=[6s_2C_0+6hs_5C_1]+6h^2s_7C_2+h^2s_2F_0+h^3s_5(6C_3+F_1)$$

$$Q_{40}=[12s_3]+4hs_1D_0+h^2s_3(4D_1+F_0)$$

$$Q_{31}=[6s_5]+3hs_7D_0+h^2s_5(3D_1+F_0)$$

$$Q_{22}=[2s_3C_0+2s_4]+2hs_1C_1+2hs_6D_0+2h^2s_3C_2+h^2s_8(2D_1+F_0)$$

$$Q_{13}=[6s_5C_0]+6hs_7C_1+hs_2D_0+6h^2s_5C_2+h^2s_5(D_1+F_0)$$

$$Q_{04}=[12s_4C_0]+12hs_6C_1+12h^2s_8C_2+h^2s_4F_0.$$

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